Inertia effects on periodic synchronization in a system of coupled oscillators

H. Hong
Department of Physics Education and Center for Theoretical Physics, Seoul National University, Seoul 151-742, Korea

M. Y. Choi and J. Yi*
Department of Physics and Center for Theoretical Physics, Seoul National University, Seoul 151-742, Korea

K.-S. Soh
Department of Physics Education and Center for Theoretical Physics, Seoul National University, Seoul 151-742, Korea
(Received 9 April 1998; revised manuscript received 27 August 1998)

We study analytically the synchronization phenomena in a set of globally coupled oscillators under external periodic driving, with emphasis on the effects of small inertia. We examine in detail both the integer and fractional mode locking present in the system with inertia, and derive the self-consistency equation for the order parameter, which reveals variation of the magnitude of the order parameter according to the external periodic driving. In particular, it is found that the inertia induces discontinuous transitions between the coherent and incoherent states. [S1063-651X(99)03701-0]

PACS number(s): 05.45.Xt, 02.50.−r, 05.70.Fh, 87.10.+e

I. INTRODUCTION

In recent years, the remarkable phenomena of *collective synchronization* in oscillatory systems, which are prevalent in physics, chemistry, biology, and social sciences, have been of much interest [1–7]. Such phenomena appear in various systems, for example, charge density waves, laser, Josephson-junction arrays, chemical reactions, and biological systems such as pacemaker cells and neurons, which may be modeled by sets of coupled nonlinear oscillators [8–12]. Due to analytic simplicity and some physical as well as biological motivation, mostly systems of globally coupled oscillators have been studied, both analytically and numerically. Among those there exist systems of oscillators, each of which is externally driven, often periodically in time. For example, many biological systems are driven by the periodic cycles of planetary motion. Further, explicit periodic driving such as laser beams, alternating currents, or microwaves may also be considered. Such a driven oscillator system is known to display characteristic mode locking, called Shapiro steps, particularly in the case of a Josephson junction. The system of coupled oscillators with external periodic driving has been investigated, to reveal periodic synchronization [13]. In that study, like most studies of the system without driving, the inertia term has not been included in the equation of motion. Namely, the inertia term has been assumed to be negligible in comparison with the damping term. The opposite limit of large inertia in the system without periodic driving has been considered recently, and the hysteresis associated with a discontinuous transition has been pointed out [14]. On the other hand, the role of small inertia terms in the system of oscillators under external periodic driving has not been addressed.

The purpose of this paper is to understand how the inertia term affects the collective synchronization in the system of coupled oscillators under periodic external driving. For this purpose, we first examine the mode locking displayed by a driven oscillator, with particular attention to the fractional locking due to the nonvanishing inertia, and investigate analytically the change of collective synchronization due to the small inertia term in the linear response to the periodic driving. The results of our theoretical analysis are as follows: In the absence of inertia only oscillators locked to the external driving contribute to the collective synchronization. In contrast, in the system with inertia unlocked oscillators as well as those locked to the external driving contribute to the collective synchronization. In particular, the inertia gives rise to hysteresis in the bifurcation diagram, and brings on discontinuous transitions. It is also found that both the time-independent and time-dependent components of the order parameter, describing the periodic synchronization, display jump discontinuities at the transition.

This paper consists of five sections: Section II introduces the driven system of coupled oscillators with small inertia terms and the characteristic mode locking displayed by the system is investigated. In Sec. III the self-consistency equation for the order parameter is derived, and discontinuous transitions between coherent and incoherent states are revealed. The periodic synchronization displayed by the system with a simple driving strength distribution is investigated in Sec. IV. Finally, in Sec. V, a brief summary is given. The Appendix presents detailed analysis of the mode locking.

II. DRIVEN SYSTEM OF COUPLED OSCILLATORS

The set of equations of motion for N coupled oscillators, the ith of which is described by its phase ϕ_i ($i=1,2,\ldots,N$), is given by

$$\mu\ddot{\phi}_i+\dot{\phi}_i+\frac{K}{N}\sum_{j=1}^{N}\sin(\phi_i-\phi_j)=\omega_i+I_i\cos\Omega t, \qquad (1)$$

*Present address: Pohang Superconductivity Center, Pohang University of Science and Technology, Pohang 790-784, Korea.

where μ denotes the magnitude of the (rotational) inertia relative to the damping. The third term on the left-hand side represents the global coupling between oscillators, with strength K/N. The first and the second terms on the right-hand-side describe the constant driving and the periodic driving on the ith oscillator, respectively. This set of equations of motion may be regarded as the mean-field version of the array of resistively and capacitively shunted junctions, which serves as a common model for describing, e.g., the dynamics of superconducting arrays [15]. In this case, the two terms on the right-hand side of Eq. (1) correspond to the combined direct and alternating current bias. The constant (dc) driving strength ω_i is distributed over the whole oscillators according to the distribution $g(\omega)$, which is assumed to be smooth and symmetric about ω_0. We may take ω_0 to be zero without loss of generality, and also assume that $g(\omega)$ is concave at $\omega=0$, i.e., $g''(0)<0$. The periodic (ac) driving strength I_i may also vary for different oscillators, while the frequency Ω of the driving is assumed to be uniform for all oscillators. Without the inertia term $(\mu=0)$, Eq. (1) precisely describes the set of equations of motion studied in Ref. [13].

Let us first consider the simple case of two coupled oscillators $(N=2)$, described by the two coupled equations

$$\mu\ddot{\phi}_1+\dot{\phi}_1+\frac{K}{2}\sin(\phi_1-\phi_2)=\omega_1+I_1\cos\,\Omega t,$$

$$\mu\ddot{\phi}_2+\dot{\phi}_2+\frac{K}{2}\sin(\phi_2-\phi_1)=\omega_2+I_2\cos\,\Omega t.$$

The above two equations can be easily decoupled by defining the relative phase $\phi\equiv\phi_1-\phi_2$; the equation of motion for ϕ reads

$$\mu\ddot{\phi}+\dot{\phi}+K\sin\,\phi=\omega+I\cos\,\Omega t, \tag{2}$$

where $\omega\equiv\omega_1-\omega_2$ and $I\equiv I_1-I_2$ represent the difference in the dc driving strength and in the ac one, respectively.

In the case of identical ac driving $(I_1=I_2)$, we have $I=0$, and Eq. (2) becomes time independent:

$$\mu\ddot{\phi}+\dot{\phi}+K\sin\,\phi=\omega, \tag{3}$$

describing a damped pendulum under constant torque or a resistively and capacitively Josephson junction driven by a direct current. With an appropriate noise term on the right-hand side, Eq. (3) would take the form of a Langevin equation. Here it is convenient to introduce the probability distribution of the phase ϕ and the velocity $\dot{\phi}$ and to consider the corresponding Fokker-Planck equation [16]. In the stationary state, we may take the average over $\dot{\phi}$, and reduce the Fokker-Planck equation to the Smoluchowski equation for the probability distribution of the phase [16,17]

$$\left(\frac{\partial V(\phi)}{\partial\phi}P(\phi)+k_BT\frac{\partial P(\phi)}{\partial\phi}\right)\left(1+\mu\frac{\partial^2V(\phi)}{\partial\phi^2}\right)\equiv C, \tag{4}$$

where C is a constant and $V(\phi)$ is the washboard potential given by

$$V(\phi)=-K\cos\,\phi-\omega\phi.$$

Since there is no noise in Eq. (3), the system is at zero (effective) temperature. We thus set $T=0$ in Eq. (4) and obtain the stationary probability distribution

$$P(\phi)=\begin{cases}\mathcal{N}\,|\omega-K\sin\,\phi|^{-1}(1+\mu K\cos\,\phi)^{-1}, & \text{for } |\omega|>K\\ \delta[\phi-\sin^{-1}(\omega/K)], & \text{for } |\omega|\leq K,\end{cases} \tag{5}$$

where $\mathcal{N}$ is the normalization constant determined by the relation $\int_0^{2\pi}P(\phi)d\phi=1$ together with the periodicity condition $P(\phi+2\pi)=P(\phi)$. In the absence of the inertia $(\mu=0)$, it is given by the well-known expression $2\pi\mathcal{N}=\sqrt{\omega^2-K^2}$. Thus for $|\omega|\leq K$, the coupling is strong enough to drive the system to the fixed point given by $\phi=\sin^{-1}(\omega/K)$, as in the case of no inertia, and we have the stationary solution, which describes the two oscillators phase locked to each other due to the coupling:

$$\phi_1=\overline{\omega}t+\frac{I_1}{\Omega(1+\mu^2\Omega^2)}(\sin\,\Omega t-\mu\Omega\cos\,\Omega t)+\phi_0,$$

$$\phi_2=\phi_1-\sin^{-1}\frac{\omega}{K}$$

with the mean frequency $\overline{\omega}\equiv(\omega_1+\omega_2)/2$ and an arbitrary constant ϕ_0. For $|\omega|>K$, on the other hand, the coupling loses in the competition with the dc driving, and the system does not possess a fixed point: ϕ increases continuously, and the two oscillators are not phase locked to each other. The average rate of increase $\langle\dot{\phi}\rangle$ is proportional to the normalization constant $\mathcal{N}$. Accordingly, $\langle\dot{\phi}\rangle/\Omega$ is in general irrational, and the system is not locked to the external driving, either.

We next consider the case in which each oscillator is driven with different ac driving strength $(I_1\neq I_2)$. In this case $(I\neq 0)$, we have a damped pendulum under periodic torque or resistively and capacitively Josephson junction driven by a combined direct and alternating current. Although Eq. (2) is not analytically tractable due to the nonlinear potential arising from the coupling between oscillators, it is known that such a system can be locked to the external driving, giving rise to steplike responses. In the absence of the inertia term $(\mu=0)$, the locking of the system is characterized by $\langle\dot{\phi}\rangle/\Omega=n$ with n integer, which is known as the (integer) Shapiro steps, particularly in the current-voltage relation of a single overdamped Josephson junction [18]. Even richer is the characteristic of the system with inertia, which allows fractional mode locking, characterized by the fractional Shapiro steps:

$$\frac{\langle \dot{\phi} \rangle}{\Omega} = \frac{p}{q} \tag{6}$$

with relatively prime integers p and q.

To investigate such mode locking in detail, we introduce an ansatz and write the phase of the oscillator in the general form

$$\phi = \phi_0 + \langle \dot{\phi} \rangle t + A_1 \sin(\Omega t + \alpha_1) + \sum_{s=1}^{q-1} \frac{q}{s} A_{s,q} \sin\left(\frac{s}{q}\Omega t + \alpha_{s,q}\right). \tag{7}$$

We first consider integer locking, where the subharmonics in Eq. (7) can be dropped. It is tedious but straightforward to obtain the solution locked on the nth step:

$$\phi \approx \phi_0 + n\Omega t + A_1 \sin(\Omega t + \alpha_1) \tag{8}$$

with $A_1 \equiv I/\Omega\sqrt{1+\mu^2\Omega^2}$ and $\alpha_1 \equiv -\tan^{-1}(\mu\Omega)$, where higher harmonics have been disregarded (see the Appendix). Note that Eq. (8) does not depend on the coupling strength K explicitly. The condition for integer locking is thus fulfilled by the dc driving lying in the range

$$n\Omega - K|J_n(A_1)| \leq \omega \leq n\Omega + K|J_n(A_1)|, \tag{9}$$

and the phase ϕ_0 in Eq. (8) is given by

$$\phi_0 = (-1)^n \sin^{-1}\left[\frac{\omega - n\Omega}{KJ_n(A_1)}\right] + n\alpha_1, \tag{10}$$

where $J_n(x)$ is the nth Bessel function, Accordingly, in the high-frequency limit $(I/\Omega \ll 1)$, the width of the nth step is given by

$$\delta\omega = 2K|J_n(A_1)| \approx \frac{K}{2^{n-1}n!}\left(\frac{I}{\Omega\sqrt{1+\mu^2\Omega^2}}\right)^n, \tag{11}$$

which shows that the inertia term tends to shrink the integer locking region, suggestive of the appearance of the additional (fractional) mode locking.

For general (fractional) locking, $\langle \dot{\phi} \rangle = (p/q)\Omega$, Eq. (7) leads to

$$\phi = \phi_0 + \frac{p}{q}\Omega t + A_1 \sin(\Omega t + \alpha_1) + \sum_{s=1}^{q-1} \frac{q}{s} A_{s,q} \sin\left(\frac{s}{q}\Omega t + \alpha_{s,q}\right), \tag{12}$$

which yields the locking condition

$$\omega = \frac{p}{q}\Omega + K\prod_{s=1}^{q-1} \sum_{\ell_s} J_{\ell_s}(qA_{s,q}/s) \times \sum_{\ell} J_{\ell}(A_1) \sin\left(\phi_0 + \ell\alpha_1 + \sum_{s=1}^{q-1} \ell_s \alpha_{s,q}\right) \tag{13}$$

with the summation performed under the constraint $p/q + \ell + \Sigma_{s=1}^{q-1} s\ell_s/q = 0$. In particular, when $p=1$, among the integer sets $(q, \ell, \Sigma_{s=1}^{q-1} s\ell_s)$ satisfying the constraint, the smallest ones give dominant contributions, e.g., $(2,-1,1)$ and $(3,-1,2)$ for the $\Omega/2$ step and the $\Omega/3$ step, respectively. Noting that the strength A_1 of the primary component is of the linear order in I, we examine the fractional mode locking $\langle \dot{\phi} \rangle/\Omega = p/q$ with $p=1$ as a linear response to the external driving. Higher fractional steps $(p \neq 1)$, on the other hand, can be shown to have the widths of higher orders in I. The two simple cases, $q=2$ $(\langle \dot{\phi} \rangle = \Omega/2)$ and $q=3$ $(\langle \dot{\phi} \rangle = \Omega/3)$, are investigated in the Appendix, yielding the step widths to the leading order,

$$\delta\omega = \begin{cases} \Omega^{-1}K^2\mu I F_{1/2}(\mu\Omega), & \text{for } q=2 \\ (9/8\Omega^2)K^3\mu I F_{1/3}(\mu\Omega), & \text{for } q=3, \end{cases} \tag{14}$$

where the precise forms of the functions $F_{1/2}(x)$ and $F_{1/3}(x)$ are presented in the Appendix. Equation (14) displays that the width $\delta\omega$ shrinks as μ is decreased, and eventually vanishes in the limit $\mu \to 0$. This manifests that finite inertia is indeed necessary for fractional locking. In the absence of the inertia term, the system exhibits only integer locking, which has been confirmed in many numerical studies.

We now return to the set of N oscillators described by Eq. (1). Collective behavior of such an N-oscillator system is conveniently described by the complex *order parameter*

$$\Psi \equiv \frac{1}{N}\sum_{j=1}^{N} e^{i\phi_j} = \Delta e^{i\theta}, \tag{15}$$

where nonvanishing Ψ indicates the appearance of synchronization. The order parameter defined in Eq. (15) allows us to reduce Eq. (1) to a *single* decoupled equation

$$\mu\ddot{\phi}_i + \dot{\phi}_i + K\Delta \sin(\phi_i - \theta) = \omega_i + I_i \cos \Omega t,$$

where Δ and θ are to be determined by imposing self-consistency. We then seek the stationary solution with constant θ, which is possible due to the symmetry of the distribution of ω_i and I_i about zero. Redefining $\phi_i - \theta$ as ϕ_i and suppressing indices, we obtain

$$\mu\ddot{\phi} + \dot{\phi} + K\Delta \sin \phi = \omega + I \cos \Omega t, \tag{16}$$

which is essentially the same as Eq. (2) except for the fact that Δ in general depends periodically on time. Note here that the order parameter Δ, which is defined in terms of the phase via Eq. (15), in turn determines the behavior of the phase via Eq. (16). The self-consistency with the behavior of the phase in Eq. (7) thus requires the expansion

$$\Delta=\Delta_0+\sum_s \Delta_s\cos(s\Omega t+\beta_s)+\sum_{p,q}\Delta_{p,q}\cos\left(\frac{p}{q}\Omega t+\beta_{p,q}\right). \tag{17}$$

Accordingly, the order parameter is composed of the time-independent dc component Δ_0, which exists regardless of the ac driving, and ac components due to the external ac driving.

The behavior of the phase governed by Eq. (16) can be examined in a manner similar to that presented in the Appendix. Namely, inserting Eqs. (7) and (17) into Eq. (16) and comparing the components term by term, we can obtain the amplitudes in Eq. (7) in terms of the components of the order parameter. (The order parameter itself can be obtained by imposing self-consistency and will be discussed in the next section.) Among them the width of the step on which oscillators are locked to the external ac driving is determined by the equation for the dc component. For example, in the case of integer locking, the resulting equation for the dc component reads

$$\omega=n\Omega+(-1)^n K\Delta_0 J_n(A_1)\sin(\phi_0-n\alpha_1), \tag{18}$$

which is simply Eq. (A5) with K replaced by $K\Delta_0$. Consequently, those oscillators with dc driving in the range

$$n\Omega-K\Delta_0|J_n(A_1)|\leq\omega\leq n\Omega+K\Delta_0|J_n(A_1)|, \tag{19}$$

which will be denoted by the notation $\omega\in S_n$, are locked to the external driving, and Eqs. (10) and (11) are still applicable simply with K replaced by $K\Delta_0$. Likewise, for fractional locking, the locking condition and the step width are given by Eqs. (13) and (14) with the appropriate replacement of K by $K\Delta_0$.

III. SELF-CONSISTENCY EQUATION FOR THE ORDER PARAMETER

In this section we derive the self-consistency equation for the order parameter, which determines the collective behavior of the system. We suppose that the periodic driving strength I is distributed according to $f(I)$, independently of the constant driving strength ω. Recalling that ϕ in Eq. (16) in fact represents $\phi-\theta$, we have the self-consistency equation

$$\Delta=\frac{1}{N}\sum_j e^{i\phi_j}=\int_{-\infty}^{\infty}dI\,f(I)\int_{-\infty}^{\infty}d\omega\,g(\omega)\langle e^{i\phi}\rangle_{\omega,I}, \tag{20}$$

where $\langle\cdots\rangle_{\omega,I}$ denotes the average in the stationary state with given ω and I.

To investigate how the inertia term affects the collective synchronization, we first consider the system without periodic driving $(I=0)$, for which Eq. (16) reads

$$\mu\ddot{\phi}+\dot{\phi}+K\Delta\sin\phi=\omega. \tag{21}$$

In this case there does not exist mode locking, and the stationary state of the system is characterized by the order parameter, which is time independent. Namely, the order parameter Δ possesses only the dc component Δ_0 in the expansion given by Eq. (17), making Eq. (21) essentially the same as Eq. (3). Accordingly, the discussion below Eq. (3) is applicable, and Eq. (21) leads to the stationary probability distribution given by Eq. (5) with K replaced by $K\Delta$. With the average taken with respect to the corresponding probability distribution, the order parameter in Eq. (20) becomes

$$\begin{aligned}\Delta&=\int_{|\omega|>K\Delta}d\omega\,g(\omega)\langle e^{i\phi}\rangle_\omega+\int_{|\omega|\leq K\Delta}d\omega\,g(\omega)\langle e^{i\phi}\rangle_\omega\\&=\left(\frac{\pi}{2}g(0)-\frac{\mu}{2}\right)K\Delta+\frac{4}{3}\mu g(0)(K\Delta)^2+\frac{\pi}{16}g''(0)(K\Delta)^3\\&\quad+O(K\Delta)^4.\end{aligned} \tag{22}$$

Note that, unlike the case of no inertia $(\mu=0)$, the oscillators with $|\omega|>K\Delta$ as well as those with $|\omega|\leq K\Delta$ contribute to the order parameter. Namely, in the system with inertia, unlocked oscillators as well as those locked to the external driving contribute to the collective synchronization. It is also of interest that the quadratic term [of order $(K\Delta)^2$], which results from the unlocked oscillators, induces hysteresis in the bifurcation diagram, since $(4/3)\mu g(0)>0$ [13]. Indeed the appearance of the hysteresis has been pointed out in the system with large inertia [14].

We now consider the effects of periodic driving, which leads to locking of the oscillators in the appropriate range. Both integer locking and fractional locking have been studied in detail in Sec. II. In particular, the ranges of the constant driving for locking and the phases of such locked oscillators allow us to compute the contributions of locked oscillators to the order parameter. The contribution of the locked oscillators to the order parameter is computed as follows:

$$\begin{aligned}&\int_{-\infty}^{\infty}dI\,f(I)\int_{-\infty}^{\infty}d\omega\,g(\omega)\langle e^{i\phi}\rangle_{\omega,I}\\&\quad=\int_{-\infty}^{\infty}dI\,f(I)\sum_{p,q}\int_{\omega\in S_{p/q}}d\omega\,g(\omega)\,\langle e^{i\phi}\rangle,\end{aligned} \tag{23}$$

where, for example, the integer locking range $S_{p/q=1}$ is given by Eq. (19) and the fractional locking ranges $S_{1/2}$ and $S_{1/3}$ by Eqs. (A10) and (A16) with K replaced by $K\Delta_0$, respectively. The phases of the oscillators in such locking ranges are given by Eqs. (8), (A6), and (A13), with the coefficients $A_{s,q}$ presented in Eqs. (A8) and (A14).

It is easy to compute the contribution from the nth integer step:

$$\begin{aligned}&\int dI\,f(I)\int_{\omega\in S_n}d\omega\,g(\omega)\langle e^{i\phi}\rangle\\&\quad=a^{(n)}K\Delta_0+b^{(n)}(K\Delta_0)^2-c^{(n)}(K\Delta_0)^3+O(K\Delta_0)^4\end{aligned} \tag{24}$$

with the coefficients

$$a^{(n)}\equiv\frac{\pi}{2}g(0)\langle J_0(A_1)\cos[A_1\sin(\Omega t+\alpha_1)]\rangle_I+\pi\sum_{n=1}^{\infty}g(2n\Omega)\cos(2n\Omega t+2n\alpha_1)\langle J_{2n}(A_1)\cos[A_1\sin(\Omega t+\alpha_1)]\rangle_I$$
$$-\pi\sum_{n=1}^{\infty}g[(2n-1)\Omega]\sin[(2n-1)(\Omega t+\alpha_1)]\langle J_{2n-1}(A_1)\sin[A_1\sin(\Omega t+\alpha_1)]\rangle_I,$$
$$b^{(n)}\equiv-\frac{4}{3}\sum_{n=1}^{\infty}(-1)^n g'(n\Omega)\sin(n\Omega t+n\alpha_1)\langle J_n^2(A_1)\cos[A_1\sin(\Omega t+\alpha_1)]\rangle_I, \tag{25}$$
$$c^{(n)}\equiv-\frac{\pi}{16}g''(0)\langle J_0^3(A_1)\cos[A_1\sin(\Omega t+\alpha_1)]\rangle_I-\frac{\pi}{8}\sum_{n=1}^{\infty}g''(2n\Omega)\cos(2n\Omega t+2n\alpha_1)\langle J_{2n}^3(A_1)\cos[A_1\sin(\Omega t+\alpha_1)]\rangle_I$$
$$+\frac{\pi}{8}\sum_{n=1}^{\infty}g''[(2n-1)\Omega]\sin[(2n-1)(\Omega t+\alpha_1)]\langle J_{2n-1}^3(A_1)\sin[A_1\sin(\Omega t+\alpha_1)]\rangle_I,$$

where $\langle\cdots\rangle_I$ denotes the average over the distribution of the ac driving strength, i.e., $\langle O\rangle_I\equiv\int_{-\infty}^{\infty}dI\,f(I)O$. In order to obtain the complete self-consistency equation for the order parameter, we should also consider the contributions from the fractional steps. The fractional locking on the p/q step in general leads to the contribution of the order of $(K\Delta_0)^q$, with frequency $(p/q)\Omega$ and amplitudes proportional to $g(p\Omega/q)$. To the order of $(K\Delta_0)^3$, we thus need to consider only the cases $q=2$ and 3. To the linear order in I, it is straightforward to observe that the symmetry of $f(I)$ about zero leads to a null contribution from the fractional locking on the 1/2 and 1/3 steps. Contributions to the higher orders in I, where the step 2/3 should also be considered, can, in principle, be computed analytically via the procedure described in the Appendix.

With the contributions from the locked oscillators as well as from the unlocked ones [in Eq. (22)] taken into account, the self-consistency equation for the order parameter in general takes the form

$$\Delta=aK\Delta_0+b(K\Delta_0)^2-c(K\Delta_0)^3+O(K\Delta_0)^4, \tag{26}$$

where the time-dependent coefficients can be expanded:

$$a=a_0+\sum_s a_s\cos(s\Omega t+\gamma_s)+\sum_{p,q}a_{p,q}\cos\left(\frac{p}{q}\Omega t+\gamma_{p,q}\right),$$
$$b=b_0+\sum_s b_s\cos(s\Omega t+\delta_s)+\sum_{p,q}b_{p,q}\cos\left(\frac{p}{q}\Omega t+\delta_{p,q}\right), \tag{27}$$
$$c=c_0+\sum_s c_s\cos(s\Omega t+\epsilon_s)+\sum_{p,q}c_{p,q}\cos\left(\frac{p}{q}\Omega t+\epsilon_{p,q}\right),$$

with the dc and ac amplitudes and phases depending on the details of the external driving.

Equation (26) describes the collective behavior of the system for given values of the parameters. The expansion given by Eq. (17) allows us to reduce Eq. (26) into the equations for the components of the order parameter [to the order of $(K\Delta_0)^3$]:

$$\Delta_0=a_0K\Delta_0+b_0(K\Delta_0)^2-c_0(K\Delta_0)^3,$$
$$\Delta_s=a_sK\Delta_0+b_s(K\Delta_0)^2\cos(\gamma_s-\delta_s)-\left[c_s\cos(\gamma_s-\epsilon_s)-\frac{b_s^2}{2a_s}\sin^2(\gamma_s-\delta_s)\right](K\Delta_0)^3, \tag{28}$$
$$\Delta_{p,q}=a_{p,q}K\Delta_0+b_{p,q}(K\Delta_0)^2\cos(\gamma_{p,q}-\delta_{p,q})-\left[c_{p,q}\cos(\gamma_{p,q}-\epsilon_{p,q})-\frac{b_{p,q}^2}{2a_{p,q}}\sin^2(\gamma_{p,q}-\delta_{p,q})\right]\times(K\Delta_0)^3,$$

with the phase

$$\beta_s=\tan^{-1}\left[\frac{a_s\sin\,\gamma_s+b_sK\Delta_0\sin\,\delta_s-c_s(K\Delta_0)^2\sin\,\epsilon_s}{a_s\cos\,\gamma_s+b_sK\Delta_0\cos\,\delta_s-c_s(K\Delta_0)^2\cos\,\epsilon_s}\right]$$

and $\beta_{p,q}$ similarly given.

We first consider the dc component Δ_0 in Eq. (28), the general behavior of which has been analyzed in Ref. [13]: In addition to the trivial solution $\Delta=0$, it allows the nontrivial solutions described by

$$\Delta_0=\Delta_\pm\equiv\frac{b_0K\pm\sqrt{(b_0^2+4c_0a_0)K^2-4c_0K}}{2c_0K^2}, \tag{29}$$

if $K\geqslant K_0\equiv 4c_0/(b_0^2+4c_0a_0)$. For simplicity, we assume that all coefficients a_0, b_0, and c_0 have positive values $(a_0,b_0,c_0>0)$. For $K<K_0$, only the null solution exists, whereas the stable nontrivial solution Δ_+ together with the unstable solution Δ_- appears via a tangent bifurcation at $K=K_0$. As K is increased further, a transcritical bifurcation arises at $K=K_c\equiv a_0^{-1}$, via which the null solution loses its stability. Thus only the nontrivial solution Δ_+ is stable for $K>K_c$. For $K_0<K<K_c$, on the other hand, both the solution Δ_+ and the null solution are possible, indicating bistability. This implies the following behavior of the system: As

K is increased from zero, the system, starting in the incoherent state, exhibits a first-order transition at $K=K_c$ into the coherent state described by the nontrivial solution, with the jump $\Delta_0^{(c)} \equiv b_0/c_0 K_c$ in the dc component Δ_0. Conversely, when K is decreased from above K_c, the system is still in the coherent state until K reaches K_0, at which the transition from the coherent state into the incoherent one occurs, with the jump $\Delta_0^{(0)} \equiv b_0/2c_0 K_0$. (See Ref. [13] for details.) In this manner the system with nonvanishing inertia exhibits hysteresis as the coupling strength is varied.

IV. PERIODIC SYNCHRONIZATION

In the system without periodic driving ($I=0$), the stationary state of the system is characterized by the time-independent order parameter. In the periodically driven system, on the other hand, the coefficients a, b, and c in Eq. (27) depend on time, and the system is expected to display periodic synchronization, characterized by periodic variation of the order parameter. In this section we study such periodic synchronization phenomena, with particular attention to the inertia effects. Since the analytical computation of the contributions from the fractional steps, which appear in higher orders in I, requires quite lengthy and complicated procedures, we mostly assume that the contributions from the integer steps are dominant over those from the fractional steps; this is the case for high-frequency driving or for small inertia and weak ac driving, and the system still displays characteristic effects of inertia on periodic synchronization. The explicit forms of the coefficients in Eq. (27) are determined for given distributions $g(\omega)$ and $f(I)$. We here suppose that the dc driving strengths are distributed in the interval $[-\omega_c, \omega_c]$, i.e., $g(\omega)\neq 0$ only for $|\omega|<\omega_c$, and take the simple $\pm I$-type distribution, $f(I)=\frac{1}{2}[\delta(I-I_0)+\delta(I+I_0)]$. Characteristic features such as periodic synchronization do not change qualitatively even if a broad distribution such as a Gaussian is used.

We first consider the simple case of high-frequency driving such that Ω is larger than $3\omega_c$, where only the $p/q=0$ step contributes to the order parameter. The coefficients in Eq. (27) in the self-consistency equation are then obtained from Eqs. (22) and (25):

$$a=\frac{\pi}{2}g(0)J_0(A_0)\cos[A_0\sin(\Omega t+\alpha_1)]-\frac{\mu}{2},$$

$$b=\frac{4}{3}\mu g(0),$$

$$c=-\frac{\pi}{16}g''(0)J_0^3(A_0)\cos[A_0\sin(\Omega t+\alpha_1)],$$

which give the amplitudes

$$\begin{aligned} a_{2n}&=\frac{\pi}{2}g(0)J_0(A_0)\,J_{2n}(A_0)-\frac{\mu}{2}\delta_{n,0},\\ b_{2n}&=\frac{4}{3}\mu g(0)\delta_{n,0},\\ c_{2n}&=-\frac{\pi}{16}g''(0)J_0^3(A_0)\,J_{2n}(A_0),\\ a_{2n-1}&=b_{2n-1}=c_{2n-1}=0 \end{aligned} \tag{30}$$

with $A_0\equiv I_0/\Omega\sqrt{1+\mu^2\Omega^2}$ and the phases $\gamma_{2n}=\epsilon_{2n}=2n\alpha_1$. Note that due to the symmetry of $f(I)$, the above amplitudes and, accordingly, the order parameter become even functions of I_0.

Equation (26) together with Eq. (30) describes the collective behavior of the system for given values of the parameters. In the absence of the inertia and ac driving ($\mu=I_0=0$), we have $\Delta=\Delta_0$ in Eq. (26) with coefficients $a=a_0=(\pi/2)g(0)$, $b=0$, and $c=c_0=-(\pi/16)g''(0)$, which reduces Eq. (26) into the self-consistency equation obtained in Ref. [8]. As pointed out already, without ac driving, the nonvanishing inertia here keeps the value of b positive, inducing hysteresis in the bifurcation diagram of the stationary order parameter $\Delta=\Delta_0$ [19]. Together with appropriate ac driving, the inertia may bring on hysteresis also in the behavior of the ac components of the order parameter. The periodic behavior of the order parameter is described by its ac components Δ_s, which are related with the dc component Δ_0 via Eq. (28). With the amplitudes given by Eq. (30), it gives the total order parameter Δ, displaying the periodicity π/Ω half the external one. Accordingly, the behavior of the dc component Δ_0 discussed in Sec. III determines that of the ac component as well. As K is increased from zero, the system thus exhibits a first-order transition at $K=K_c$ from the incoherent state to the coherent one, where both dc and ac components of the order parameter acquire nonzero values. In particular, the ac component Δ_{2n} ($n\geqslant 1$) as well as the dc component display jump discontinuities at $K=K_c$, given by $\Delta_{2n}^{(c)}=a_{2n}(b_0/c_0)-c_{2n}(b_0/c_0)^3$. Similarly, when K is decreased from above K_c, the first-order transition occurs at $K=K_0$, with the jump in the ac component $\Delta_{2n}^{(0)}=a_{2n}(b_0/2c_0)-c_{2n}(b_0/2c_0)^3$. Thus for sufficiently large K the system is in the coherent state, exhibiting periodic synchronization.

Figure 1 shows the behavior of the total order parameter Δ at given time t, given by Eqs. (26) and (30), as the coupling strength K is varied. Both (a) the case of zero inertia ($\mu=0$) and (b) that of nonvanishing one ($\mu=0.05$) are displayed, at time $t=5$. The driving strength and frequency have been chosen to be $I_0=2$ and $\Omega=5$ while the dc driving strengths to follow a semicircle distribution with unit radius. In the absence of inertia ($\mu=0$), as shown in (a), coherent behavior emerges via a continuous transition at $K=K_c\approx 1.0842$. On the other hand, (b) displays the first-order transitions in the system with nonvanishing inertia ($\mu=0.05$), appearing at $K_0\approx 1.1038$ and at $K_c\approx 1.1089$, and associated hysteresis. Note that K_c is increased from the value without inertia, indicating that the inertia tends to suppress coherence. The corresponding time evolution of the order param-

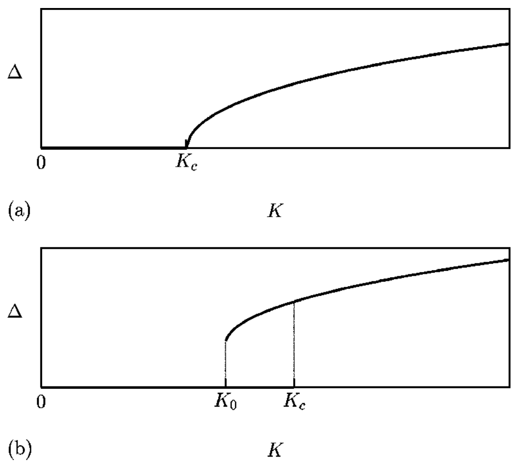

FIG. 1. Behavior of the order parameter with the coupling strength, (a) in the absence of inertia ($\mu=0$) and (b) in the presence of inertia ($\mu\neq 0$). In (a) a pitchfork bifurcation is shown to occur at $K=K_c$, while (b) displays a tangent bifurcation at K_0 and a transcritical bifurcation at K_c. The dotted lines indicate discontinuous jumps between the coherent and incoherent states.

eter for $K=1.11$ is shown in Fig. 2, where the periodicity $\pi/\Omega=\pi/5$ can be observed. Comparison of (a) and (b) indeed reveals that the inertia tends to reduce the value of the order parameter, thus suppressing coherence in the system.

We next consider the case of small inertia ($\mu\ll 1$) and weak ac driving ($I_0\ll 1$). In this case the fractional steps give contributions of the order of μI_0^2, and may be neglected in the self-consistency equation to the order of I_0^2, leading to Eq. (27) in the form

$$a=a_0+a_2\cos(2\Omega t+2\alpha_1)+O(\mu I_0^2),$$

$$b=b_0+b_1\sin(\Omega t+\alpha_1)+O(\mu I_0^2),$$

$$c=c_0+c_2\cos(2\Omega t+2\alpha_1)+O(\mu I_0^2),$$

with the amplitudes

$$a_0\equiv\frac{\pi}{2}g(0)-\frac{\mu}{2}-\frac{\pi I_0^2}{4\Omega^2}g(0)-\frac{\pi I_0^2}{4\Omega^2}g(\Omega),$$

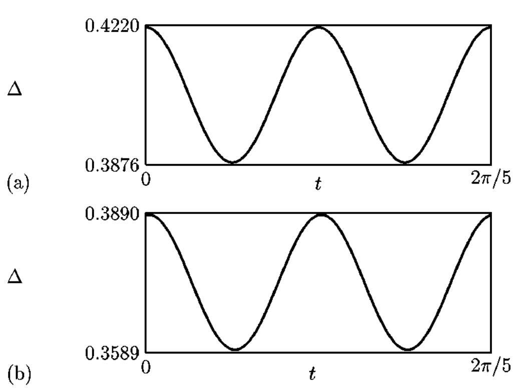

FIG. 2. Periodic synchronization in the system with $K=1.11$, $I=2.0$, and $\Omega=5.0$, for (a) $\mu=0$ and (b) $\mu=0.05$. Comparison of the two reveals that the inertia tends to suppress synchronization.

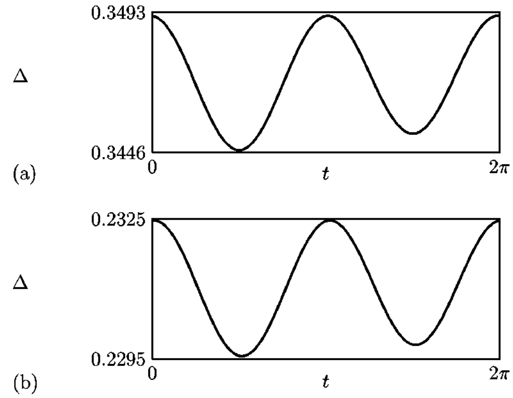

FIG. 3. Periodic synchronization in the system with $K=1.68$, $I=0.1$, and $\Omega=1.0$, for (a) $\mu=0$ and (b) $\mu=0.05$. It is again observed that the inertia tends to suppress the synchronization.

$$a_2\equiv\frac{\pi I_0^2}{8\Omega^2}g(0)+\frac{\pi I_0^2}{8\Omega^2}g(2\Omega)+\frac{\pi I_0^2}{4\Omega^2}g(\Omega),$$

$$b_0\equiv\frac{4}{3}\mu g(0),$$

$$b_1\equiv\frac{I_0^2}{3\Omega^2}g'(\Omega),$$

$$c_0\equiv-\frac{\pi}{16}g''(0)+\frac{\pi I_0^2}{16\Omega^2}g''(0),$$

$$c_2\equiv-\frac{\pi I_0^2}{64\Omega^2}g''(0).$$

The total order parameter Δ thus reads, to the order of I_0^2,

$$\Delta=\Delta_0+b_1(K\Delta_0)^2\sin(\Omega t-\tan^{-1}\mu\Omega)+[a_2(K\Delta_0)-c_2(K\Delta_0)^3]\cos(2\Omega t-2\tan^{-1}\mu\Omega), \quad (31)$$

displaying the fundamental periodicity $2\pi/\Omega$ together with its harmonics π/Ω. Similarly to the previous case of high-frequency driving, the total order parameter Δ also exhibits a first-order transition between the incoherent state and the coherent one; at the transition both dc and ac components of the order parameter display jump discontinuities.

As the coupling strength K is varied, the order parameter Δ given by Eq. (31) displays behaviors qualitatively similar to those for high-frequency driving, shown in Fig. 1. As an example, we choose a Gaussian distribution with unit variance for the dc driving and $I_0=0.1$ and $\Omega=1$ for the ac driving. In the system with $\mu=0$ this leads to a continuous transition at $K_c\approx 1.6087$; for $\mu=0.05$ first-order transitions at $K_0\approx 1.6698$ and at $K_c\approx 1.6762$. Figure 3 shows the corresponding time evolution for $K=1.68$, which displays the fundamental periodicity $2\pi/\Omega=2\pi$ as well as the harmonics π/Ω. Again the suppressing effects of the inertia on synchronization is manifested.

It is also of interest to note the possibility that the amplitudes of the ac components can be of the same order as that of the dc component for appropriate ranges of the parameters. For example, in Eq. (30) the ratio $a_2/a_0 = J_2(A_0)/\{J_0(A_0)-[\pi g(0)J_0(A_0)]^{-1}\mu\}$ may become of the order unity for $I_0 \gtrsim 1$. Then the order parameter Δ may sometimes vanish during its periodic time evolution, displaying synchronization-desynchronization cycles in time. Such cycles have been observed in the numerical simulations of the system without inertia [20].

V. SUMMARY

We have studied analytically the synchronization phenomena in a set of globally coupled oscillators under external periodic driving. To understand how the inertia term affects the collective synchronization, we have investigated the change of collective synchronization due to the small inertia term in the linear response to the periodic driving. In the absence of inertia, only oscillators locked to the external driving contribute to the collective synchronization. In contrast, in the system with inertia, unlocked oscillators as well as those locked to the external driving contribute to the collective synchronization. The resulting self-consistency equation for the order parameter reveals variation of the magnitude of the order parameter according to the external periodic driving. The dependence of the characteristic behavior of the order parameter on the inertia as well as on the external driving has been examined: The inertia tends to suppress coherence and affects the details of the behavior, such as the nature of transitions and the period of the system. In particular, it has been found that the inertia gives rise to hysteresis in the bifurcation diagram and induces discontinuous transitions between the coherent and incoherent states. Here only analytical results have been presented, and it would be of interest to confirm these results via large-scale numerical simulations.

ACKNOWLEDGMENTS

We are grateful to K. Park for useful discussions, and acknowledge partial support from the Ministry of Education of Korea through the BSRI Program, and from the Korea Science and Engineering Foundation through the SRC Program. H.H. also thanks the Korea Research Foundation for financial support.

APPENDIX

1. Integer locking: $\langle \dot{\phi} \rangle = n\Omega$

Inserting the ansatz

$$\phi(t) = \phi_0 + n\Omega t + \sum_s A_s \sin(s\Omega t + \alpha_s) \tag{A1}$$

into Eq. (2), we obtain

$$\mu \sum_s s^2\Omega^2 A_s \sin(s\Omega t + \alpha_s) - \sum_s s\Omega A_s \cos(s\Omega t + \alpha_s) + I\cos\Omega t = n\Omega - \omega + K\prod_s \sum_{\ell_s} J_{\ell_s}(A_s)\sin\Phi_{\ell_s}(t),$$

where $J_l(x)$ is the lth Bessel function and $\Phi_{\ell_s}(t) \equiv \phi_0 + \Sigma_s \ell_s \alpha_s + (n + \Sigma_s s\ell_s)\Omega t$. The integers ℓ_s satisfying $\Sigma_s s\ell_s = -n$ contribute to the dc component in Φ_{ℓ_s}:

$$\omega - n\Omega = K\prod_s \sum_{\ell_s}{}' J_{\ell_s}(A_s)\sin\left(\phi_0 + \sum_s \ell_s\alpha_s\right), \tag{A2}$$

where the prime in the summation represents the constraint $\Sigma_s s\ell_s = -n$. This gives an estimation of the dc driving strength ω corresponding to the integer locking. Further, the amplitude A_s and phase α_s of the component with frequency $s\Omega$ can be determined from the equation

$$\mu s^2\Omega^2 A_s \sin(s\Omega t + \alpha_s) - s\Omega A_s\cos(s\Omega t + \alpha_s) + \delta_{s,1} I\cos s\Omega t = K\prod_{s=1}^{N}\sum_{\ell_s^+} J_{\ell_s^+}(A_s)\sin\left(\phi_0 + s\Omega t + \sum_{s=1}^{N}\ell_s^+\alpha_s\right) + K\prod_{s=1}^{N}\sum_{\ell_s^-} J_{\ell_s^-}(A_s)\sin\left(\phi_0 - s\Omega t + \sum_{s=1}^{N}\ell_s^-\alpha_s\right), \tag{A3}$$

with integers ℓ_s^+ and ℓ_s^- satisfying $\Sigma_s s\ell_s^+ = s - n$ and $\Sigma_s s\ell_s^- = -s - n$, respectively. In Eq. (A3) the ac driving with frequency Ω gives the contribution, independent of K, to the amplitude A_1. On the other hand, the leading order contribution to A_s with $s > 1$ is obviously given by K. When K is sufficiently small compared with the driving frequency and amplitude, Eq. (A3) with $s=1$ takes the approximate form

$$\mu\Omega^2 A_1 \sin(\Omega t + \alpha_1) - A_1\Omega\cos(\Omega t + \alpha_1) + I\cos\Omega t \approx KJ_{1-n}(A_1)\sin[\phi_0 + \Omega t - (n-1)\alpha_1] - KJ_{1+n}(A_1)\sin[\phi_0 - \Omega t - (n+1)\alpha_1],$$

which yields A_1 and α_1 to the zeroth order in K:

$$A_1 = \frac{I}{\Omega\sqrt{1+\mu^2\Omega^2}},$$

$$\alpha_1 = -\tan^{-1}(\mu\Omega).$$

This gives the (locked) phase of the oscillator on the nth step:

$$\phi \approx \phi_0 + n\Omega t + \frac{I}{\Omega\sqrt{1+\mu^2\Omega^2}}\sin[\Omega t - \tan^{-1}(\mu\Omega)], \tag{A4}$$

which is just Eq. (8). Neglecting the higher-order contributions from A_s with $s>1$ in Eq. (A2), we further have

$$\omega = n\Omega + (-1)^n K J_n(A_1)\sin(\phi_0 - n\alpha_1), \tag{A5}$$

which leads to the range of the dc driving strength given by Eq. (9) and the constant ϕ_0 in Eq. (10).

2. Fractional locking

$\langle\dot{\phi}\rangle = \Omega/2$. The simple trial solution

$$\phi(t) = \phi_0 + \frac{\Omega}{2}t + A_1\sin(\Omega t + \alpha_1) + 2A_{1,2}\sin\left(\frac{\Omega}{2}t + \alpha_{1,2}\right) \tag{A6}$$

leads the equation of motion in Eq. (16) to take the form

$$\mu\Omega^2 A_1\sin(\Omega t + \alpha_1) - A_1\Omega\cos(\Omega t + \alpha_1) + I\cos\Omega t + \frac{\mu\Omega^2}{2}A_{1,2}\sin\left(\frac{\Omega}{2}t + \alpha_{1,2}\right) - A_{1,2}\Omega\cos\left(\frac{\Omega}{2}t + \alpha_{1,2}\right) = K\sum_{n,\ell} J_n(2A_{1,2})J_\ell(A_1)\sin\Phi_{n,\ell}(t) + \frac{\Omega}{2} - \omega,$$

where $\Phi_{n,\ell}(t) \equiv \phi_0 + n\alpha_{1,2} + \ell\alpha_1 + (1/2 + n/2 + \ell)\Omega t$. Thus the terms with $\ell + 1/2 = -n/2$ contribute to the dc component, and give the equation for the dc component

$$\frac{\Omega}{2} - \omega - KJ_1(2A_{1,2})J_0(A_1)\sin(\phi_0 - \alpha_{1,2}) - KJ_1(2A_{1,2})J_1(A_1)\sin(\phi_0 + \alpha_{1,2} - \alpha_1) + O(K^3, I^2) = 0. \tag{A7}$$

On the other hand, the amplitude and phase of the component with frequency $\Omega/2$ is determined by the equation

$$-\frac{\mu\Omega^2}{2}A_{1,2}\sin\left(\frac{\Omega}{2}t + \alpha_{1,2}\right) + A_{1,2}\Omega\cos\left(\frac{\Omega}{2}t + \alpha_{1,2}\right) + KJ_0(2A_{1,2})J_0(A_1)\sin\left(\frac{\Omega}{2}t + \phi_0\right) + KJ_0(2A_{1,2})J_1(A_1)\sin\left(\frac{\Omega}{2}t + \alpha_1 - \phi_0\right) + O(K^2, I^2) = 0,$$

which, upon expanding the Bessel function $J_n(z)$ for $z \ll 1$, yields

$$A_{1,2}\Omega\cos\alpha_{1,2} = \frac{K}{\sqrt{1+\mu^2\Omega^2/4}}\left[\frac{\mu\Omega}{2}\cos\phi_0 - \sin\phi_0 + \frac{I}{4\Omega}P(\mu\Omega;\phi_0)\right],$$

$$A_{1,2}\Omega\sin\alpha_{1,2} = \frac{K}{\sqrt{1+\mu^2\Omega^2/4}}\left[\frac{\mu\Omega}{2}\sin\phi_0 + \cos\phi_0 - \frac{I}{4\Omega}Q(\mu\Omega;\phi_0)\right] \tag{A8}$$

with $P(x;\phi_0) \equiv (1+x^2)^{-1}[3x\cos\phi_0 + (2-x^2)\sin\phi_0]$ and $Q(x;\phi_0) \equiv (1+x^2)^{-1}[3x\sin\phi_0 + (x^2-2)\cos\phi_0]$. Equation (A7) combined with Eq. (A8) leads to the following expression:

$$\omega = \omega_{1/2} - \frac{\mu I}{2\Omega}K^2 F_{1/2}(\mu\Omega)\sin(2\phi_0 - \alpha_1), \tag{A9}$$

where

$$\omega_{1/2} \equiv \frac{\Omega}{2} + \frac{K^2}{\Omega(1+\mu^2\Omega^2/4)},$$

$$F_{1/2}(x) \equiv (1+x^2)^{-1/2}\left(1 + \frac{x^2}{4}\right)^{-1},$$

and we have used $A_1 = I/\Omega\sqrt{1+\mu^2\Omega^2}$. Equation (A9) yields the range of the dc driving strength corresponding to the $\Omega/2$ locking:

$$\omega_{1/2} - \frac{\mu I}{2\Omega}K^2 F_{1/2}(\mu\Omega) \leqslant \omega \leqslant \omega_{1/2} + \frac{\mu I}{2\Omega}K^2 F_{1/2}(\mu\Omega), \tag{A10}$$

and the constant phase ϕ_0 in Eq. (A9) is given by

$$\phi_0 = \frac{1}{2}\sin^{-1}y - \frac{\alpha_1}{2} \tag{A11}$$

with

$$y \equiv -\frac{2\Omega}{\mu I K^2 F_{1/2}(\mu\Omega)}(\omega - \omega_{1/2}).$$

Equation (A10) leads to the step width for $\langle\dot{\phi}\rangle = \Omega/2$:

$$\delta\omega = \Omega^{-1}K^2\mu I F_{1/2}(\mu\Omega), \tag{A12}$$

which is presented in Eq. (14).

$\langle\dot{\phi}\rangle = \Omega/3$. In a similar manner, we take the trial solution displaying $\langle\dot{\phi}\rangle = \Omega/3$,

$$\phi(t) = \phi_0 + \frac{\Omega}{3}t + A_1\sin(\Omega t + \alpha_1) + 3A_{1,3}\sin\left(\frac{\Omega}{3}t + \alpha_{1,3}\right) + \frac{3}{2}A_{2,3}\sin\left(\frac{2\Omega}{3}t + \alpha_{2,3}\right), \tag{A13}$$

and obtain the coefficients

$$A_{1,3}\Omega\cos\alpha_{1,3}=\left(1+\frac{\mu^2\Omega^2}{9}\right)^{-1}\left[K\left(\frac{\mu\Omega}{3}\cos\phi_0-\sin\phi_0\right)+K^2\left(\frac{2\mu\Omega}{3}B(\mu\Omega)-C(\mu\Omega)\right)\right],$$

$$A_{1,3}\Omega\sin\alpha_{1,3}=\left(1+\frac{\mu^2\Omega^2}{9}\right)^{-1}\left[K\left(\frac{\mu\Omega}{3}\sin\phi_0+\cos\phi_0\right)+K^2\left(\frac{\mu\Omega}{3}C(\mu\Omega)+B(\mu\Omega)\right)\right],$$

$$A_{2,3}\Omega\cos\alpha_{2,3}=\frac{K}{2\Omega}I(1+\mu^2\Omega^2)^{-1}\left(1+\frac{4\mu^2\Omega^2}{9}\right)^{-1}\left[\frac{5\mu\Omega}{3}\cos\phi_0-\left(\frac{2\mu^2\Omega^2}{3}-1\right)\sin\phi_0\right]$$
$$+K^2\left(1+\frac{4\mu^2\Omega^2}{9}\right)^{-1}\left(\frac{2\mu\Omega}{3}L(\mu\Omega)-M(\mu\Omega)\right),$$

$$A_{2,3}\Omega\sin\alpha_{2,3}=\frac{K}{2\Omega}I(1+\mu^2\Omega^2)^{-1}\left(1+\frac{4\mu^2\Omega^2}{9}\right)^{-1}\left[-\frac{5\mu\Omega}{3}\sin\phi_0+\left(1-\frac{2\mu^2\Omega^2}{3}\right)\cos\phi_0\right]$$
$$+K^2\left(1+\frac{4\mu^2\Omega^2}{9}\right)^{-1}\left(\frac{2\mu\Omega}{3}M(\mu\Omega)+L(\mu\Omega)\right), \tag{A14}$$

where

$$\frac{B(x)}{D(x)}\equiv\left(\frac{x}{3}-\frac{11}{27}x^3\right)\cos 2\phi_0-\left(1+\frac{19}{9}x^2+\frac{10}{27}x^4\right)\sin 2\phi_0,$$

$$\frac{C(x)}{D(x)}\equiv\left(-\frac{x}{3}+\frac{11}{27}x^3\right)\sin 2\phi_0-\left(1+\frac{19}{9}x^2+\frac{10}{27}x^4\right)\cos 2\phi_0,$$

with

$$D(x)\equiv\frac{3I}{8\Omega^2}(1+x^2)^{-1}\left(1+\frac{x^2}{9}\right)^{-1}\left(1+\frac{4x^2}{9}\right)^{-1}$$

and

$$L(x)\equiv\frac{3}{2\Omega}\left(1+\frac{x^2}{9}\right)^{-1}\left(\frac{x}{3}\cos 2\phi_0-\sin 2\phi_0\right),$$

$$M(x)\equiv\frac{3}{2\Omega}\left(1+\frac{x^2}{9}\right)^{-1}\left(\frac{x}{3}\sin 2\phi_0+\cos 2\phi_0\right).$$

The dc driving strength corresponding to the $\Omega/3$ locking is also obtained:

$$\omega=\omega_{1/3}-\frac{9K^3}{16\Omega^2}\mu IF_{1/3}(\mu\Omega)\sin(3\phi_0+\eta) \tag{A15}$$

with

$$\omega_{1/3}\equiv\frac{\Omega}{3}+\frac{3K^2}{2\Omega(1+\mu^2\Omega^2/9)},$$

$$F_{1/3}(x)\equiv(1+x^2)^{-1}\left(1+\frac{x^2}{9}\right)^{-2}\left(1+\frac{4x^2}{9}\right)^{-1}\times\left(1+\frac{31x^2}{9}+\frac{85x^4}{27}+\frac{61x^6}{81}+\frac{4x^8}{81}\right)^{1/2},$$

$$\eta(x)\equiv\tan^{-1}\left[\left(x+\frac{5x^3}{3}+\frac{2x^5}{9}\right)\left(\frac{x^2}{3}-\frac{x^4}{9}\right)^{-1}\right].$$

This gives the locking condition for $\langle\dot{\phi}\rangle=\Omega/3$:

$$\omega_{1/3}-\frac{9K^3}{16\Omega^2}\mu IF_{1/3}(\mu\Omega)\leqslant\omega\leqslant\omega_{1/3}+\frac{9K^3}{16\Omega^2}\mu IF_{1/3}(\mu\Omega). \tag{A16}$$

Likewise the constant phase ϕ_0 in Eq. (A14) is given by

$$\phi_0=\frac{1}{3}\sin^{-1}\left[\frac{16\Omega^2}{9\mu IK^3F_{1/3}(\mu\Omega)}(\omega-\omega_{1/3})\right]-\frac{\eta}{3}. \tag{A17}$$

We thus obtain the width of the step $\langle\dot{\phi}\rangle=\Omega/3$

$$\delta\omega=\frac{9}{8\Omega^2}K^3\mu IF_{1/3}(\mu\Omega), \tag{A18}$$

which is given in Eq. (14).

[1] For a list of references, see A.T. Winfree, *The Geometry of Biological Time* (Springer-Verlag, New York, 1980); Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer-Verlag, Berlin, 1984); L. Glass and M.C. Mackey, *From Clocks to Chaos: The Rhythm of Life* (Princeton University, Princeton, 1988).

[2] H. Haken, *Advanced Synergetics* (Springer-Verlag, Berlin, 1983); R. Serra, M. Andretta, M. Compiani, and G. Zanarini,

Introduction to the Physics of Complex Systems (Pergamon, Oxford, 1986).

[3] T.D. Clark, Phys. Lett. A **27**, 585 (1968); Phys. Rev. B **8**, 137 (1973).

[4] S.P. Benz, M. Rzchowski, M. Tinkham, and C.J. Lobb, Phys. Rev. Lett. **64**, 693 (1990); M.Y. Choi, Phys. Rev. B **46**, 564 (1992); S. Kim, B.J. Kim, and M.Y. Choi, *ibid.* **52**, 13 536 (1995).

[5] L. Fabiny, P. Colet, and R. Roy, Phys. Rev. A **47**, 4287 (1993); R. Roy and K.S. Thornburg, Jr., Phys. Rev. Lett. **72**, 2009 (1994); J.F. Heagy, T.L. Caroll, and L.M. Pecora, Phys. Rev. E **50**, 1874 (1994).

[6] T.J. Walker, Science **166**, 891 (1969); M.K. McClintock, Nature (London) **229**, 244 (1971); D.C. Michaels, E.P. Maytyas, and J. Jalife, Circ. Res. **61**, 704 (1987); J. Buck, Q. Rev. Biol. **63**, 265 (1988); R.D. Traub, R. Miles, and R.K.S. Wong, Science **243**, 1319 (1989).

[7] R. Eckhorn *et al.*, Biol. Cybern. **60**, 121 (1988); C.M. Gray and W. Singer, Proc. Natl. Acad. Sci. USA **86**, 1698 (1989).

[8] Y. Kuramoto, in *Proceedings of the International Symposium on Mathematical Problems in Theoretical Physics*, edited by H. Araki (Springer-Verlag, New York, 1975); Y. Kuramoto and I. Nishikawa, J. Stat. Phys. **49**, 569 (1987); Y. Kuramoto, Physica D **50**, 15 (1991).

[9] H. Daido, Prog. Theor. Phys. **77**, 622 (1987); Phys. Rev. Lett. **68**, 1073 (1992).

[10] S.H. Strogatz, C.M. Marcus, R.M. Westervelt, and R.E. Mirollo, Physica D **36**, 23 (1989); J.W. Swift, S.H. Strogatz, and K. Wiesenfeld, *ibid.* **55**, 239 (1992); S.H. Strogatz, R.E. Mirollo, and P.C. Matthews, Phys. Rev. Lett. **68**, 2730 (1992); S. Watanabe and S.H. Strogatz, *ibid.* **70**, 2391 (1993); K. Wiesenfeld and J.W. Swift, Phys. Rev. E **51**, 1020 (1995); K. Wiesenfeld, P. Colet, and S.H. Strogatz, Phys. Rev. Lett. **76**, 404 (1996).

[11] A. Arenas and C.J. Perez Vicente, Europhys. Lett. **26**, 79 (1994).

[12] K. Park and M.Y. Choi, Phys. Rev. E **52**, 2907 (1995); Phys. Rev. B **56**, 387 (1997); K. Park, S.W. Rhee, and M.Y. Choi, Phys. Rev. E **57**, 5030 (1998).

[13] M.Y. Choi, Y.W. Kim, and D.C. Hong, Phys. Rev. E **49**, 3825 (1994).

[14] H. Tanaka, A.J. Lichtenberg, and S. Oishi, Phys. Rev. Lett. **78**, 2104 (1997).

[15] W.C. Stewart, Appl. Phys. Lett. **22**, 277 (1968); V. Ambegaokar and B.I. Halperin, Phys. Rev. Lett. **22**, 1364 (1969): J.S. Chung, K.H. Lee, and D. Stroud, Phys. Rev. B **40**, 6570 (1989).

[16] H. Risken, *The Fokker-Planck Equation: Methods of Solution and Applications* (Springer-Verlag, Berlin, 1989).

[17] T. Schneider, E.P. Stoll, and R. Morf, Phys. Rev. B **18**, 1417 (1978).

[18] S. Shapiro, Phys. Rev. Lett. **11**, 80 (1963); S. Shapiro, A.R. Janus, and S. Holly, Rev. Mod. Phys. **36**, 223 (1964); K.H. Lee, D. Stroud, and J.S. Chung, Phys. Rev. Lett. **64**, 962 (1990); J.U. Free *et al.*, Phys. Rev. B **41**, 7267 (1990); M.Y. Choi, *ibid.* **46**, 564 (1992).

[19] Note that the sign of b here has been defined in the way opposite to that in Ref. [13].

[20] B.G. Yoon, M.S. Chung, and M.Y. Choi, Sae Mulli [New Physics] **38**, 146 (1998).